Series SGN

MATHEMATICS

Paper & Solutions

SET-1

Code : 65/1

Time : 3 Hrs. **Max. Marks : 100**

General Instruction :

(i) All questions are compulsory.

(ii) The question paper consists of **29** questions divided into four section A, B, C and D. Sections A comprises of **4** questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 3 questions of four marks each and 3 questions of six marks each. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is **not** permitted. You may ask for logarithmic tables, if required.

SECTION - A

Question numbers 1 to 4 carry 1 mark each

1. Find the value of $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$

Sol. $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$

$$\frac{\pi}{3} - \frac{5\pi}{6} = \frac{2\pi - 5\pi}{6} = \frac{-3\pi}{6} = \frac{-\pi}{2}$$

2. If the matrix $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is skew symmetric, find the values of 'a' and 'b'.

Sol. $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$

For skew symmetric matrix

$A^T = -A$

$$\begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -a & 3 \\ -2 & 0 & 1 \\ -b & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a & 3 \\ -2 & 0 & 1 \\ -b & -1 & 0 \end{bmatrix}$$

$b = 3$ (On comparing LHS & RHS)

$a = -2$

3. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$.

Sol. Magnitude of two vectors $\vec{a}$ & $\vec{b}$ are same $|\vec{a}| = |\vec{b}|$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\frac{a}{2} = |\vec{a}||\vec{a}|\cos 60°$$

$$|\vec{a}|^2 = \frac{9}{2}\times 2 = 9$$

$$|\vec{a}| = 3 = |\vec{b}|$$

4. If a ∗ b denotes the larger of 'a' and 'b' and if a o b = (a ∗ b) + 3, then write the value of (5) o (10), where ∗ and o are binary operations.

Sol. a o b = (a * b) + 3

5 o 10 = (5 * 10) + 3

= 10 + 3 = 13

SECTION - B

Question numbers 5 to 12 carry 2 marks each

5. Prove that : $3\sin^{-1} x = \sin^{-1}(3x - 4x^3),\ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Sol. $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$

Let $\sin^{-1} x = \theta$

$x = \sin\theta$

$\therefore\ \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\sin 3\theta = 3x - 4x^3$

Case I When $\frac{-1}{2} \le x \le \frac{1}{2}$

$$\frac{-1}{2} \le \sin\theta \le \frac{1}{2}$$

$$\frac{-\pi}{6} \le \theta \le \frac{\pi}{6}$$

$$\frac{-\pi}{2} \le 3\theta \le \frac{\pi}{2}$$

Also $\frac{-1}{2} \le x \le \frac{1}{2} \Rightarrow -1 \le 3x - 4x^3 \le 1$

$$\sin 3\theta = 3x - 4x^3$$

$$3\theta = \sin^{-1}(3x - 4x^3)$$

$$3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$$

6. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$.

Sol. $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{14-12}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

To Prove $2A^{-1} = 9I - A$

LHS = $2A^{-1}$

$$= 2 \times \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

RHS = $9I - A$

$$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

LHS = RHS

7. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x.

Sol. $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right) = y$ (Let)

$$y = \tan^{-1}\left(\frac{2\cos^2 \frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)\right]$$

$$y = \frac{\pi}{2} - \frac{x}{2}$$

$$\frac{dy}{dx} = 0 - \frac{1}{2} = \frac{-1}{2}$$

8. The total cost C(x) associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.

Sol. $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000.$

marginal cost (MC) = $\frac{dC(x)}{dx}$

$= (0.005)(3x^2) - 0.02\,(2x) + 30$

When x = 3 $\quad MC = 0.005\,(3 \times 9) - 0.02(2 \times 3) + 30$

$= 30.015$

9. Evaluate : $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\,dx$

Sol. $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\,dx$

$\int \frac{2\cos^2 x - 1 + 2(1 - \cos^2 x)}{\cos^2 x}\,dx$

$\int \left(2 - \frac{1}{\cos^2 x} + \frac{2}{\cos^2 x} - 1\right) dx$

$\int \left(1 + \frac{1}{\cos^2 x}\right) dx$

$x + \int \sec^2 x\,dx$

$x + \tan x + C$

10. Find the differential equation representing the family of curves $y = a\,e^{bx+5}$, where a and b are arbitrary constants.

Sol. $y = a\,e^{bx+5}$

Take log on both sides

$\log y = \log (a\,e^{bx+5})$

$\log y = \log a + \log(e^{bx+5})$

Differentiate both sides w.r.t. x

$\frac{1}{y}\frac{dy}{dx} = 0 + \frac{d}{dx}(bx + 5)$

$\frac{1}{y}\frac{dy}{dx} = b$

$\Rightarrow \frac{1}{y}\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

$\Rightarrow \frac{d^2y}{dx^2} + y\left(\frac{dy}{dx}\right)^2 = 0$

i.e. the required differential eq^n.

11. If θ is the angle between two vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$, find sin θ.

Sol. $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\vec{a}\cdot\vec{b} = (\hat{i}-2\hat{j}+3\hat{k})\cdot(3\hat{i}-2\hat{j}+\hat{k})$

$= 3 + 4 + 3$

$= 10$

$10 = \left(\sqrt{1+4+9}\right)\left(\sqrt{9+4+1}\right)\cos\theta$

$\cos\theta = \dfrac{10}{\sqrt{14}\sqrt{14}}$

$\cos\theta = \dfrac{10}{14}$

$\sin\theta = \sqrt{1-\cos^2\theta}$

$= \sqrt{1-\dfrac{25}{49}} = \sqrt{\dfrac{49-25}{49}}$

$= \dfrac{\sqrt{24}}{7} = \dfrac{2\sqrt{6}}{7}$

12. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Sol. Let F : no. of red die is less than 4.

E : sum of no is 8

E = {(2, 6) (3, 5) (4, 4) (5, 6) (6, 2)}

$\Rightarrow P(E) = \dfrac{5}{36}$

F = {(1, 1) (2, 1) (3, 1) .... (6, 1)

(1, 2) (2, 2) .... (6, 2)

(1, 3) (2, 3) .... (6, 3)}

$P(F) = \dfrac{18}{36}$

Also $E \cap F = \{(5, 3)\ (6, 2)\} \Rightarrow P(E \cap F) = \dfrac{2}{36}$

$P(E/F) = \dfrac{P(E\cap F)}{P(F)} = \dfrac{2/36}{18/36}$

$= \dfrac{1}{9}$

SECTION - C

Question numbers 13 to 23 carry 4 marks each

13. Using properties of determinants, prove that $\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9\,(3xyz + xy + yz + zx)$.

Sol. $\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix}$

$$xyz\begin{vmatrix} \frac{1}{x} & \frac{1}{x} & \frac{1}{x}+3 \\ \frac{1}{y}+3 & \frac{1}{y} & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z}+3 & \frac{1}{z} \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= xyz\begin{vmatrix} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 & \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 & \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3 \\ \frac{1}{y}+3 & \frac{1}{y} & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z}+3 & \frac{1}{z} \end{vmatrix}$$

$$= (xyz)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{y}+3 & \frac{1}{y} & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z}+3 & \frac{1}{z} \end{vmatrix}$$

$C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$

$$= xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{y}+3 & -3 & -3 \\ \frac{1}{z} & 3 & 0 \end{vmatrix}$$

$= (yz + zx + xy + 3xyz)\ 1\ (0 + 9)$

$= 9(3xyz + xy + yz + zx) = \text{RHS}$

14. If $(x^2 + y^2)^2 = xy$, find $\frac{dy}{dx}$.

OR

If $x = a\,(2\theta - \sin 2\theta)$ and $y = a\,(1 - \cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$.

Sol. $(x^2 + y^2)^2 = xy$

$2(x^2 + y^2)\,[2x + 2y\frac{dy}{dx}] = x\frac{dy}{dx} + y$

$4x\,(x^2 + y^2) + 4y(x^2 + y^2)\,\frac{dy}{dx} = x\frac{dy}{dx} + y$

$4x(x^2 + y^2) - y = \frac{dy}{dx}\,[x - 4y\,(x^2 + y^2)]$

$$\frac{dy}{dx} = \frac{4x(x^2+y^2)-y}{x-4y(x^2+y^2)}$$

OR

$x = a(2\theta - \sin 2\theta)$, $y = a(1 - \cos 2\theta)$

$\frac{dx}{d\theta} = a(2 - 2\cos 2\theta)$, $\frac{dy}{d\theta} = a(0 + 2\sin 2\theta)$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$
$$= \frac{2a\sin 2\theta}{2a(1-\cos 2\theta)}$$
$$\left(\frac{dy}{dx}\right)_{(\theta=\pi/3)} = \frac{\sin 2\pi/3}{1-\cos 2\pi/3}$$
$$= \frac{\sin(\pi-\pi/3)}{1-\cos(\pi-\pi/3)}$$
$$= \frac{\sin\pi/3}{1+\cos\pi/3} = \frac{\sqrt{3}/2}{1+1/2}$$
$$= \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

15. If y = sin (sin x), prove that $\frac{d^2y}{dx^2} + \tan x\,\frac{dy}{dx} + y\cos^2 x = 0$.

Sol. y = sin(sinx)

$$\frac{dy}{dx} = \cos(\sin x)\cos x.$$
$$\frac{d^2y}{dx^2} = \cos(\sin x)(-\sin x) + \cos^2 x[-\sin(\sin x)]$$
$$\frac{d^2y}{dx^2} = -\sin x\cos(\sin x) - \cos^2 x\sin(\sin x)$$
$$\text{LHS} = \frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x$$
$= -\sin x\cos(\sin x) - \cos^2 x\sin(\sin x) + \tan x\cos x\cos(\sin x) + \cos^2 x\sin x(\sin x)$

$= -\sin x\cos(\sin x) + \frac{\sin x}{\cos x}\cos x\cos(\sin x)$

$= -\sin x\cos(\sin x) + \sin x\cos(\sin x)$

= 0 = RHS

16. Find the equations of the tangent and the normal, to the curve $16x^2 + 9y^2 = 145$ at the point (x_1, y_1), where $x_1 = 2$ and $y_1 > 0$.

OR

Find the intervals in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is

(a) Strictly increasing, (b) Strictly decreasing.

Sol. $16x^2 + 9y^2 = 145$ .....(1)

since (x_1, y_1) lies on (1)

$16x_1^2 + 9y_1^2 = 145$

$16(2)^2 + 9y_1^2 = 145 \quad \Rightarrow y_1 = 3$

(2, 3)

$16x^2 + 9y^2 = 145$

Diff. w.r.t. to x

$32x + 18y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-16x}{9y}$

$\left(\frac{dy}{dx}\right)_{(2,3)} = \frac{-16}{9} \times \frac{2}{3} = \frac{-32}{27}$

Eq^n of tangent

$(y - 3) = m(x - 2)$

$(y - 3) = \frac{-32}{27}(x - 2)$

$27y - 81 = -32x + 64$

$32x + 27y = 145$

Eq^n of normal $(y - y_1) = \frac{-1}{m}(x - x_1)$

$(y - 3) = \frac{27}{32}(x - 2)$

$32y - 96 = 27x - 54$

$27x - 32y + 42 = 0$

OR

$f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$

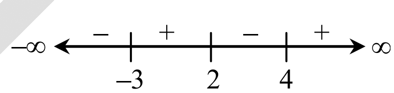
$f'(x) = \frac{4x^3}{4} - 3x^2 - 10x + 24.$

$f'(x) = 0$

$x^3 - 3x^2 - 10x + 24 = 0$

$(x - 2)(x^2 - x - 12) = 0$

$(x - 2)(x - 4)(x + 3) = 0$



increasing in interval $(-3, 2) \cup (4, \infty)$

decreasing in interval $(-\infty, -3) \cup (2, 4)$

17. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question?

Sol. Let the length, width & height of the open tank be x, x & y units

Volume = $x^2 y$

Total surface area = $x^2 + 4xy$

$S = x^2 + 4x\left(\frac{V}{x^2}\right)$

$\frac{dS}{dx} = 2x - \frac{4V}{x^2} = 0$

$2x^3 = 4V$

$2x^3 = 4x^2y$

$x = 2y$

$\frac{d^2S}{dx^2} = 2 + \frac{8V}{x^3}$

$= 2 + \frac{8V}{8y^3} = 2 + \frac{V}{y^3} > 0$

Hence S is minimum when x = 2y ie the depth (height) of the tank is half of the width

18. Find $\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)} dx$

Sol. $\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)} dx$

put sinx = t

cos x dx = dt

$\int \frac{2dt}{(1-t)(1+t^2)}$

$\frac{2}{(1-t)(1+t^2)} = \frac{A}{1-t} + \frac{Bt+C}{1+t^2}$

$2 = A(1 + t^2) + (Bt + C)(1 - t)$

put $1 - t = 0$ | $2 = A(2)$

$t = 1$ | $A = 1$

Comparing coefficients of t^2 & t

$t^2 \rightarrow A + (-B) = 0$

$B = A$

$B = 1$

$t \rightarrow B - C = 0$

$B = C = 1$

$\int \left(\frac{1}{1-t} + \frac{t+1}{t^2+1}\right) dt$

$\frac{\log(1-t)}{-1} + \int \frac{t}{t^2+1} dt + \int \frac{1}{t^2+1} dt$

$-\log(1 - \sin x) + \frac{1}{2}\log(t^2 + 1) + \tan^{-1} t + C$

$-\log(1 - \sin x) + \frac{1}{2}\log(\sin^2 x + 1) + \tan^{-1}(\sin x) + C$

19. Find the particular solution of the differential equation $e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dy = 0$, given that $y = \frac{\pi}{4}$ when $x = 0$.

OR

Find the particular solution of the differential equation $\frac{dy}{dx} + 2y \tan x = \sin x$, given that $y = 0$ when $x = \frac{\pi}{3}$.

Sol. $e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dy = 0$

$e^x \tan y\, dx = (e^x - 2) \sec^2 y\, dy$

$$\frac{dy}{dx} = \frac{e^x \tan y}{e^x \sec^2 y - 2\sec^2 y}$$

$$\frac{dx}{dy} = \frac{e^x \sec^2 y - 2\sec^2 y}{e^x \tan y}$$

$$\frac{dx}{dy} = \frac{\sec^2 y}{\tan y} - \frac{2\sec^2 y}{\tan y} e^{-x}$$

$$\frac{dx}{dy} = \frac{\sec^2 y}{\tan y}\left[1 - 2e^{-x}\right]$$

$$\int \frac{\sec^2 y}{\tan y} dy = \int \frac{1}{1 - 2e^{-x}} dx$$

$\tan y = t$

$\sec^2 y\, dy = dt$

$$\int \frac{dt}{t} = \int \frac{e^x}{e^x - 2} dx$$

$e^x - 2 = u$

$e^x\, dx = du$

$\log t = \log u + \log C$

$\log(\tan y) = \log(e^x - 2)C$

$\tan y = C(e^x - 2)$

put $y = \frac{\pi}{4}$, $x = 0$ $\qquad \tan \frac{\pi}{4} = C(1 - 2)$

$C = -1$

$\tan y = -(e^x - 2)$

OR

$$\frac{dy}{dx} + (2\tan x)y = \sin x$$

$\text{IF} = e^{\int 2\tan x}$

$= e^{2\log \sec x}$

$= \sec^2 x.$

$$y.\sec^2 x = \int \sec^2 x \cdot \sin x\, dx + C$$

$$= \int \frac{\sin x}{\cos^2 x} dx + C$$

$$= \int \tan x \sec x\, dx + C$$

$y \sec^2 x = \sec x + C$

20. Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d} \cdot \vec{a} = 21$.

Sol. $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$

$\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$

$\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$

To find vector $\vec{d}$, such that

$\vec{d} \cdot \vec{c} = 0$

$\vec{d} \cdot \vec{b} = 0$

& $\vec{d} \cdot \vec{a} = 21$

Let $\vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$

$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (3\hat{i} + \hat{j} - \hat{k}) = 0$

$3x + y - z = 0$ .....(1)

$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} - 4\hat{j} + 5\hat{k}) = 0$

$x - 4y + 5z = 0$ .....(2)

$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (4\hat{i} + 5\hat{j} - \hat{k}) = 21$

$4x + 5y - z = 21$ ......(3)

eq^n (1) × 4 + eq^n (2)

$12x + 4y - 4z = 0$

$\underline{x - 4y + 5z = 0}$

$13x + z = 0$ ....(4)

eq^n (2) × 5 + eq^n (3) × 4

$5x - 20y + 25z = 0$

$\underline{16x + 20y - 4z = 84}$

$21x + 21z = 84$

$21(x + z) = 84$

$x + z = 4$ ......(5)

eq. (4) – (5)

$12x = -4$

$x = \frac{-4}{12} = \frac{-1}{3}$

$z = 4 - x$

$z = 4 + \frac{1}{3} = \frac{13}{3}$

Put x & z in (1)

$3x + y - z = 0$

$3 \times \left(\frac{-1}{3}\right) + y - \frac{13}{3} = 0$

$y = \frac{13}{3} + 1$

$y = \frac{16}{3}$

$\vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{d} = \frac{-1}{3}\hat{i} + \frac{16}{3}\hat{j} + \frac{13}{3}\hat{k}$

21. Find the shortest distance between the lines

$\vec{r} = (4\hat{i} - \hat{j}) + \lambda\,(\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu\,(2\hat{i} + 4\hat{j} - 5\hat{k})$.

Sol. $\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$

$\vec{a}_1 = 4\hat{i} - \hat{j} \qquad \vec{b}_1 = \hat{i} + 2\hat{j} - 3\hat{k}$

$\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$

$\vec{a}_2 = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b}_2 = 2\hat{i} + 4\hat{j} - 5\hat{k}$

$$\text{S.D.} = \left|\frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$

$= \hat{i}(-10 + 12) - \hat{j}(-5 + 6) + \hat{k}(4 - 4)$

$= 2\hat{i} - \hat{j}$

$$\text{S.D.} = \left|\frac{(-3\hat{i} + 0\hat{j} + 2k)\cdot(2\hat{i} - \hat{j})}{\sqrt{4+1}}\right|$$

$$= \left|\frac{-6}{\sqrt{5}}\right| = \frac{6}{\sqrt{5}}$$

22. Suppose a girl throws a die. If she gets 1 or 2, she tosses a coin three times and notes the number of tails. If she gets 3, 4, 5 or 6, she tosses a coin once and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3, 4, 5 or 6 with the die?

Sol. Let E_1 be the event that the girl. Gets 1 or 2 on the roll

$P(E_1) = \frac{2}{6} = \frac{1}{3}$

Let E_2 be the event that the girl gets 3, 4, 5 or 6 on the roll $P(E_2) = \frac{4}{6} = \frac{2}{3}$

Let A be event that she obtained exactly one tails

If she tossed a coin 3 times & exactly 1 tail shows then [HTH, HHT, THH} = 3

$P(A/E_1) = 3/8$

$P(A/E_2) = 1/2$ (If she tossed a coin only once & exactly 1 shows)

$$P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{2}{3}}{\frac{1}{2}\times\frac{2}{3} + \frac{3}{8}\times\frac{1}{3}} = \frac{8}{11}$$

23. Two numbers are selected at random (without replacement) from the first five positive integers. Let X denote the larger of the two numbers obtained. Find the mean and variance of X.

Sol. The first five positive integers are 1, 2, 3, 4, 5 we select two positive numbers in 5 × 4 = 20 ways

Out of these two no. are selected at random & let X denote larger of the two no.

X can be 2, 3, 4 or 5

P(X = 2) = P(larger no. is 2) = {(1, 2) and (2, 1)}

$= \frac{2}{30}$

$P(X = 3) = \frac{4}{30}$

$P(X = 4) = \frac{6}{30}$

$P(X = 5) = \frac{8}{30}$

Mean = E(X) = $2 \times \frac{2}{30} + 3 \times \frac{4}{30} + 4 \times \frac{6}{30} + 5 \times \frac{8}{30}$

$= \frac{4+12+24+40}{30}$

$= \frac{80}{30}$

Variance = $2^2 \times \frac{2}{30} + 3^2 \times \frac{4}{30} + 4^2 \times \frac{6}{30} + 5^2 \times \frac{8}{30}$

$= \frac{8+36+96+200}{30}$

$= \frac{340}{30} = \frac{34}{3}$

SECTION - D

Question numbers 24 to 29 carry 6 marks each

24. Let A = {x ∈ Z : 0 ≤ x ≤ 12}. Show that
R = {(a, b) : a, b ∈ A, |a – b| is divisible by 4} is an equivalence relation.
Find the set of all elements related to 1. Also write the equivalence class [2].

OR

Show that the function f : R → R defined by $f(x) = \frac{x}{x^2+1}$, ∀ x ∈ R is neither one-one nor onto. Also, if g : R → R is defined as g(x) = 2x – 1, find fog(x).

Sol. R = {(a, b) : a, b ∈ A, |a – b| is divisible by 4}

Reflexivity : for any a ∈ A
|a – a| = 0, which is divisible by 4
(a, a) ∈ R.
So, R is reflexive.

Symmetry : Let (a, b) ∈ R
|a – b| is divisible by 4
⇒ |b – a| is also divisible by 4
So R is symmetry

Transitive : Let (a, b) ∈ R & (b, c) ∈ R
|a – b| is divisible by 4
|a – b | = 4λ
a – b = ± 4λ ....(1)
|b – c| is divisible by 4
|b – c| = 4μ
b – c = ±4μ ....(2)

Add (1) & (2)

$a - b + b - c = \pm 4(\lambda + \mu)$

$a - c = \pm 4(\lambda + \mu)$

$\Rightarrow (a, c) \in R$

So, Transitive

Hence, R is reflexive, Symmetry & Transitive so it is an equivalence relation

Let x be an element of A such that $(x, 1) \in R$, then

$|x - 1|$ is divisible by 4

$x - 1 = 0, 4, 8, 12.$

$\Rightarrow x = 1, 5, 9$

Hence, the set of all element of A which are related to 1 in {1, 5, 9}

OR

$f(x) = \frac{x}{x^2+1}$

for one-one f(x) = f(y)

$\frac{x}{x^2+1} = \frac{y}{y^2+1}$

$xy^2 + x = yx^2 + y$

$xy(y - x) = y - x$

$xy = 1$

$x = \frac{1}{y}$

$x \neq y$

So not one-one

for onto $f(x) = y$

$\frac{x}{x^2+1} = y$

$x = yx^2 + y$

$x^2y + y - x = 0$

x cannot be express in y so not onto

As $g(x) = 2x - 1$

$fog(x) = f[g(x)] = f(2x - 1) = \frac{2x-1}{(2x-1)^2+1}$

$= \frac{2x-1}{4x^2-4x+2}$

25. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Use it to solve the system of equations

$2x - 3y + 5z = 11$

$3x + 2y - 4z = -5$

$x + y - 2z = -3.$

OR

Using elementary row transformations, find the inverse of the matrix

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$

Sol. $|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix} = 2(-4+4) + 3(-6+4) + 5(3-2)$

$\therefore |A| = 0 - 6 + 5 = -1 \neq 0$

Now $A_{11} = 0; A_{12} = 2; A_{13} = 1$

$A_{21} = -1; A_{22} = -9; A_{23} = -5$

$A_{31} = 2; A_{32} = 23; A_{33} = 13$

$$A_{ij} = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}$$

$$\therefore \text{Adj } A = [A_{ij}]^T = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}A}{|A|} = -\frac{1}{1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

$$\text{Now } \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$\Rightarrow A X = B$

$$\Rightarrow X = A^{-1} B = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}\begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore$ x = 1; y = 2 and z = 3

OR

A = IA (Inverse of matrix)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 3R_3$

$R_2 \to R_2 - R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & -3 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$

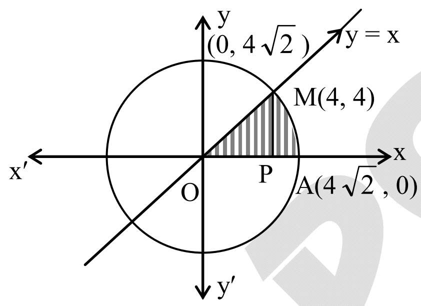
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} A$$

$$A^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

26. Using integration, find the area of the region in the first quadrant enclosed by the x-axis, the line y = x and the circle $x^2 + y^2 = 32$.

Sol. $x^2 + y^2 = 32$

$x^2 + y^2 = \left(\sqrt{32}\right)^2 = \left(4\sqrt{2}\right)^2$



For coordinate of M put y = x in $x^2 + y^2 = 32$

$2x^2 = 32$

$x^2 = 16$

$x = \pm 4$

M(4, 4)

Required Area = area of shaded region

= area of OMA

= area OMP + area MPA

$$= \int_0^4 y_1 dx + \int_4^{4\sqrt{2}} y_2 dx$$

$$= \int_0^4 x\, dx + \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - x^2}\, dx$$

$$= \left(\frac{x^2}{2}\right)_0^4 + \left[\frac{x}{2}\sqrt{(4\sqrt{2})^2 - x^2} + \frac{(4\sqrt{2})^2}{2}\sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}}$$

$$= \frac{16}{2} + \left(\frac{4\sqrt{2}}{2}\sqrt{(4\sqrt{2})^2 - (4\sqrt{2})^2} + \frac{32}{2}\sin^{-1} 1\right)$$

$$- \left(\frac{4}{2}\sqrt{(4\sqrt{2})^2 - 4^2} + \frac{32}{2}\sin^{-1}\frac{1}{\sqrt{2}}\right)$$

$$= 8 + (2\sqrt{2}\,(0) + 16 \times \frac{\pi}{2}) - (2 \times 4 + 16 \times \frac{\pi}{4})$$

$= 8 + 8\pi - 8 - 4\pi$

$= 4\pi$

27. Evaluate : $\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x} dx$

OR

Evaluate $\int_1^3 (x^2 + 3x + e^x)\, dx$,

as the limit of the sum.

Sol. $\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x} dx$

$\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9[1 - (\sin x - \cos x)^2]} dx$

$\int_0^{\pi/4} \frac{\sin x + \cos x}{25 - 9(\sin x - \cos x)^2} dx$

sin x – cos x = t

(cos x + sin x) dx = dt

$\int_{-1}^{0} \frac{dx}{25 - 9t^2}$

$\int_{-1}^{0} \frac{1}{9\left(\frac{25}{9} - t^2\right)} dt$

$\frac{1}{9}\int_{-1}^{0} \frac{1}{\left(\frac{5}{3}\right)^2 - t^2}$

$\frac{1}{9}\left[\frac{1}{2 \times \frac{5}{3}} \log\left|\frac{\frac{5}{3} + t}{\frac{5}{3} - t}\right|\right]_{-1}^{0}$

$\frac{1}{9} \times \frac{1}{\frac{10}{3}}\left(\log\left|\frac{\frac{5}{3}}{\frac{5}{3}}\right|\right) - \log\left|\frac{\frac{2}{3}}{\frac{8}{3}}\right|$

$\frac{1}{30}\left[\log 1 - \log \frac{1}{4}\right]$

$\frac{1}{30} \log 4 = \frac{1}{15} \log 2$

OR

$\int_1^3 (x^2 + 3x + e^x)\, dx$

$= \lim_{h \to 0} h[f(1) + f(1 + h) + f(1 + 2h) + \ldots + f(1 + 2(n - 1)h)]$

$= \lim_{h \to 0} h[(1 + 3 + e) + ((1 + h)^2 + 3(1 + h) + e^{1+h}) + ((1 + 2h)^2 + 3(1 + 2h) + e^{1 + 2h}) + \ldots]$

$= \lim_{h \to 0} h[4 + e + (1 + h^2 + 2h + 3 + 3h + e^{1+h}) + (1 + 4h^2 + 4h + 3 + 6h + e^{1 + 2h}) + \ldots]$

$$= \lim_{h\to 0} h[4 + e + (4 + h^2 + 5h + e^{1+h}) + (4 + 4h^2 + 10h + e^{1+2h}) + \ldots]$$

$$= \lim_{h\to 0} h[4n + e(1 + e^h + e^{2h} + \ldots) + h^2[1^2 + 2^2 + \ldots] + 5h[1 + 2 + \ldots.)]$$

$$= \lim_{h\to 0} h\left[4n + e\left(\frac{1(e^{hn}-1)}{e^h - 1}\right) + h^2\left(\frac{n(n-1)(2n-1)}{6}\right) + 5h\left(\frac{n(n-1)}{2}\right)\right]$$

$$= \lim_{h\to 0} h\left[4n + e\left(\frac{e^{nh}-1}{e^h-1}\right) + \frac{h^2n^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + \frac{5hn^2}{2}\left(1-\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[4n + e\left(\frac{e^{\frac{n\times 2}{n}}-1}{e^{2/n}-1}\right) + \frac{4}{n^2}\frac{n^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + \frac{5}{2}\times n^2\times\frac{2}{n}\left(1-\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[4n + e\left(\frac{e^2-1}{e^{2/n}-1}\right) + \frac{4n}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + n5\left(1-\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty} 2\left[4 + \frac{e\,(e^2-1)}{n\,e^{2/n}-1} + \frac{2}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + 5\left(1-\frac{1}{n}\right)\right]$$

$$= 8 + e(e^2 - 1) + \frac{4}{3} + 5$$

$$= \frac{24 + 4 + 15}{3} + e(e^2 - 1)$$

$$= \frac{43}{3} + e(e^2 - 1)$$

28. Find the distance of the point (– 1, – 5, – 10) from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda\,(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}\,.\,(\hat{i} - \hat{j} + \hat{k}) = 5$.

Sol. Equation of line is

$\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$

Equation of plane is $\vec{r}\cdot(\hat{i} - \hat{j} + \hat{k}) = 5$

Now combined equation

$[(2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})](\hat{i} - \hat{j} + \hat{k}) = 5$

$[(2 + 3\lambda)\hat{i} + (-1 + 4\lambda)\hat{j} + (2 + 2\lambda)\hat{k}]\cdot(\hat{i} - \hat{j} + \hat{k}) = 5$

$2 + 3\lambda - (-1 + 4\lambda) + 2 + 2\lambda = 5$

$\lambda + 5 = 5$

$\lambda = 0$

So, equation of line is

$\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + 0(3\hat{i} + 4\hat{j} + 2\hat{k})$

$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k}$

Let the point of intersection be (x, y, z)

So, $\vec{r} = x\hat{i} + y\hat{j} + 2\hat{k}$

$\therefore$ x = 2, y = –1, z = 2

$\therefore$ point of intersection is (2, – 1, 2)

Distance b/w (2, – 1, 2) & (– 1, – 5, – 10)

$= \sqrt{(-1-2)^2 + (-5+1)^2 + (-10-2)^2}$

$= \sqrt{9+16+144} = 13$

29. A factory manufactures two types of screws A and B, each type requiring the use of two machines, an automatic and a hand-operated. It takes 4 minutes on the automatic and 6 minutes on the hand-operated machine to manufacture a packet of screws 'A'. While it takes 6 minutes on the automatic and 3 minutes on the hand-operated machine to manufacture a packet to screws 'B'. Each machine is available for at most 4 hours on any day. The manufacturer can sell a packet of screws 'A' at a profit of 70 paise and screws 'B' at a profit of ₹ 1. Assuming that he can sell all the screws he manufactures, how many packets of each type should the factory owner produce in a day in order to maximize his profit? Formulate the above LPP and solve it graphically and find the maximum profit.

Sol. Let the number of package of screw A = x

Number of packages of screw B = y

Item	Number	Machine A	Machine B	Profit
Screw A	x	4 minutes	6 minutes	To paise = .7 Rs
Screw B	y	6 minutes	3 minutes	Rs. 1
Max time Available		4 hours = 240 min	4 hours = 240 minutes	

Automated Machine	Hand operated machine
Works for screw A $\rightarrow$ 4 min	Works on screw A $\rightarrow$ 6 min
Works on screw B $\rightarrow$ 6 min	Works on screw B $\rightarrow$ 3 min
$\therefore$ $4x + 6y \le 240$	$\therefore$ $6x + 3y \le 240$
$2x + 3y \le 120$	$2x + y \le 80$
$x, y \ge 0$	$x, y \ge 0$

Now max Z = 0.7 x + y

s.t. $2x + 3y \le 120$

$2x + y \le 80$

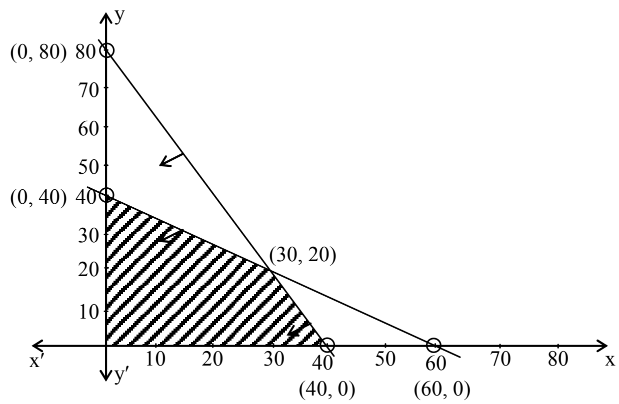
$x, y \ge 0$

$2x + 3y \le 120$

x	0	60
y	40	0

$2x + y \le 80$

x	0	40
y	80	0



Corner Points	Value of Z
(0, 40)	40
(30, 20)	41
(40, 0)	28

Hence, profit will be maximum, if the company produces,

30 packages of screw A

20 packages of screw B

Maximum Profit = Rs 41